

# DIRECT SUMS OF TRACE MAPS AND SELF-ADJOINT EXTENSIONS

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ABSTRACT. We give a simple criterion so that a countable infinite direct sum of trace (evaluation) maps is a trace map. An application to the theory of self-adjoint extensions of direct sums of symmetric operators is provided; this gives an alternative approach to results recently obtained by Malamud-Neidhardt and Kostenko-Malamud using regularized direct sums of boundary triplets.

## 1. INTRODUCTION

We begin with a simple example. Let  $\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}$  be the Laplace-Beltrami operator on the two-dimensional cylinder  $\mathbb{M}_0 := \mathbb{R}_+ \times \mathbb{T}$  with respect to the flat Riemannian metric  $g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Its minimal realization with domain  $C_c^\infty(\mathbb{M}_0)$  is symmetric and negative as a linear operator in the Hilbert space  $L^2(\mathbb{M}_0) = L^2(\mathbb{R}_+) \otimes L^2(\mathbb{T})$ . We denote its Friedrichs' self-adjoint extension by  $\Delta_0^D$ ; it corresponds to imposing Dirichlet boundary conditions at the boundary  $\mathbb{T}$ , i.e.  $\mathcal{D}(\Delta_0^D) = \{u \in H^2(\mathbb{M}_0) : \lim_{x \downarrow 0} u(x, \theta) = 0\}$ . Here  $H^2(\mathbb{M}_0)$  is the usual Sobolev-Hilbert space of order two. Let us denote by  $H^s(\mathbb{T})$  the (fractional) Sobolev-Hilbert space of square-integrable functions  $f$  on the 1-dimensional torus  $\mathbb{T}$  such that  $\sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{f}_k|^2 < +\infty$ , where  $\hat{f}_k$  is the usual Fourier coefficient  $\hat{f}_k := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ik\theta} f(\theta) d\theta$ . Then  $\gamma_0 : \mathcal{D}(\Delta_0^D) \rightarrow H^{\frac{1}{2}}(\mathbb{T})$ , the unique continuous linear map which on regular functions acts by

$$\gamma_0 u(\theta) = \lim_{x \downarrow 0} \frac{\partial u}{\partial x}(x, \theta),$$

is a concrete example of what we call an *abstract trace map* (see the next section), i.e.  $\gamma_0$  is continuous (w.r.t. graph norm), surjective and its kernel is dense in  $L^2(\mathbb{M}_0)$ . By partial Fourier transform with respect to the angular variable one gets

$$L^2(\mathbb{M}_0) = \bigoplus_{k \in \mathbb{Z}} L^2(\mathbb{R}_+), \quad \Delta_0^D = \bigoplus_{k \in \mathbb{Z}} d_k^2,$$

where

$$d_k^2 : \mathcal{D}(d_k^2) \subset L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+), \quad d_k^2 f := f'' - k^2 f,$$

$$\mathcal{D}(d_k^2) = \mathcal{D}_0 := \{f \in L^2(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}_+}) : f'' \in L^2(\mathbb{R}_+), f(0) = 0\}.$$

On  $\mathcal{D}_0$  one can define the trace map

$$\hat{\gamma}_0 : \mathcal{D}_0 \rightarrow \mathbb{C}, \quad \hat{\gamma}_0 f := f'(0),$$

which is bounded, surjective and with a kernel dense in  $L^2(\mathbb{R}_+)$ . Moreover  $\hat{\gamma}_0$  is bounded uniformly in  $k \in \mathbb{Z}$  w.r.t. the graph norm of  $d_k^2$ , and so the infinite direct sum

$$(1.1) \quad \bigoplus_{k \in \mathbb{Z}} \hat{\gamma}_0 : \mathcal{D}(\bigoplus_{k \in \mathbb{Z}} d_k^2) \rightarrow \ell^2(\mathbb{Z}).$$

is a well defined bounded operator. Since  $\gamma_0$  corresponds to  $\bigoplus_{k \in \mathbb{Z}} \hat{\gamma}_0$  by partial Fourier transform, (1.1) does not define a trace map since it is not surjective: its range space is the strict subspace of  $\ell^2(\mathbb{Z})$  defined by

$$h^{\frac{1}{2}}(\mathbb{Z}) := \left\{ \{s_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{k \in \mathbb{Z}} |k| |s_k|^2 < +\infty \right\} \simeq H^{\frac{1}{2}}(\mathbb{T}).$$

This simple example shows that an infinite direct sum of trace maps can fail to be a trace map: the direct sum of the range spaces can be different from the range space of the sum.

In Section 2 we provide a simple criterion which selects the right range space in order that the direct sum of trace maps is a trace map. Such a simple criterion uses an hypothesis involving the boundedness of operator-valued sequences obtained composing the trace maps with their right inverses (see (2.1)). Such a hypothesis seems a very strong one (indeed that allows an easy proof), however we show that always there exist right inverses such that (2.1) holds true (see Lemma 2.3).

In Section 3 we give an application to self-adjoint extensions of direct sums of symmetric operators and provide a couple of examples. We obtain that the methods here presented permit to obtain results equivalent to the ones recently obtained in [8] and [7] using regularized boundary triplets (see Remark 3.5).

In Example 1 we determine the trace space for the evaluation map  $f \mapsto \{f'(x_n)\}_{n \in \mathbb{N}}$  acting on functions  $f \in H^2(\mathbb{R} \setminus X) \cap H_0^1(\mathbb{R} \setminus X)$  where  $X = \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ ,  $x_n < x_{n+1}$ . In this case Theorem 2.1 easily implies that the range space is a weighted  $\ell^2$ -space with weight  $w_n = d_n^{-1}$ , where  $d_n := x_{n+1} - x_n$ . By Theorem 3.2 such a trace map can be used to define one-dimensional Schrödinger operators with  $\delta$  and  $\delta'$  interaction supported on the discrete set  $X$ , thus providing a construction alternative to the one presented in [7].

In Example 2 we show that our criterion easily gives the correct trace space  $H^{\frac{1}{2}}(\mathbb{T})$  for the example provided at the beginning. Then we point out that the same criterion allows to prove that  $H^s(\mathbb{T})$ ,  $s = \frac{1}{2} - \frac{\alpha}{1+\alpha}$ , is (isomorphic to) the defect space of  $\Delta_\alpha^{\min}$ ,  $-1 < \alpha < 1$ , the minimal realization of the Laplace-Beltrami operator  $\Delta_\alpha := \frac{\partial^2}{\partial x^2} - \frac{\alpha}{x} \frac{\partial}{\partial x} + x^{2\alpha} \frac{\partial^2}{\partial \theta^2}$  corresponding to the degenerate/singular Riemannian metric  $g_\alpha(x, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2\alpha} \end{pmatrix}$ . We refer to the papers [3] and [4] for the almost-Riemannian geometric considerations leading to the study of  $\Delta_\alpha$  and to [12] for the classification of all self-adjoint extensions of  $\Delta_\alpha^{\min}$ .

## 2. DIRECT SUMS OF ABSTRACT TRACE MAPS

Let  $\mathcal{H}_k$ ,  $k \in \mathbb{Z}$ , be a sequence of Hilbert spaces, with scalar product  $\langle \cdot, \cdot \rangle_k$  and corresponding norm  $\| \cdot \|_k$ . On each  $\mathcal{H}_k$  we consider a self-adjoint operator

$$A_k : \mathcal{D}(A_k) \subset \mathcal{H}_k \rightarrow \mathcal{H}_k$$

and we denote by  $\mathcal{H}_{(k)}$  the Hilbert space made by  $\mathcal{D}(A_k)$  equipped with a scalar product  $\langle \cdot, \cdot \rangle_{(k)}$  giving rise to a norm  $\| \cdot \|_{(k)}$  equivalent to the graph one.

Let  $\mathfrak{h}_k$ ,  $k \in \mathbb{Z}$ , be a sequence of auxiliary Hilbert spaces with scalar product  $[\cdot, \cdot]_k$  and corresponding norm  $|\cdot|_k$ .

Let

$$\tau_k : \mathcal{H}_{(k)} \rightarrow \mathfrak{h}_k, \quad k \in \mathbb{Z},$$

be a sequence of *abstract trace maps*, i.e.  $\tau_k$  is a linear, continuous and surjective map such that its kernel  $\mathcal{K}(\tau_k)$  is dense in  $\mathcal{H}_{(k)}$ . Since  $\tau_k$  is continuous and surjective there exists a linear continuous right inverse

$$\iota_k : \mathfrak{h}_k \rightarrow \mathcal{H}_{(k)}, \quad \tau_k \iota_k = 1$$

(see e.g. [2, Proposition 1, Section 6, Chapter 4]). Since  $\tau_k$  is surjective,  $\iota_k$  is injective and so we can define a new scalar product on  $\mathfrak{h}_k$  by

$$[\phi_k, \psi_k]_{(k)} := [\iota_k^* \iota_k \phi_k, \psi_k]_k \equiv \langle \iota_k \phi_k, \iota_k \psi_k \rangle_{(k)}.$$

It is immediate to check that  $\mathfrak{h}_k$  is complete w.r.t. the norm

$$|\phi_k|_{(k)} := \|\iota_k \phi_k\|_{(k)} \equiv |(\iota_k^* \iota_k)^{1/2} \phi_k|_k.$$

Let us denote by  $\mathfrak{h}_{(k)}$  the Hilbert space given by  $\mathfrak{h}_k$  equipped with the scalar product  $[\cdot, \cdot]_{(k)}$ . We pose

$$\mathcal{H} := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k, \quad \mathcal{H}_\circ := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_{(k)},$$

$$\mathfrak{h} := \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_k, \quad \mathfrak{h}_\circ := \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_{(k)}$$

with corresponding norms  $\| \cdot \|$ ,  $\| \cdot \|_\circ$ ,  $|\cdot|$ ,  $|\cdot|_\circ$ .

We denote by  $\|\cdot\|$  the operator norm of bounded linear operators.

**Theorem 2.1.** *Let  $\iota_k$  be a linear continuous right inverse of  $\tau_k$  and suppose that*

$$(2.1) \quad \sup_{k \in \mathbb{Z}} \|\iota_k \tau_k\| < +\infty.$$

*Then the linear map*

$$\tau : \mathcal{H}_\circ \rightarrow \mathfrak{h}_\circ, \quad \tau\left(\bigoplus_{k \in \mathbb{Z}} v_k\right) := \bigoplus_{k \in \mathbb{Z}} (\tau_k v_k)$$

*is an abstract trace map, i.e. is continuous, surjective and its kernel  $\mathcal{K}(\tau)$  is dense in  $\mathcal{H}$ .*

*Proof.* (continuity) Let  $v = \bigoplus_{k \in \mathbb{Z}} v_k \in \mathcal{H}_\circ$ . Then

$$\begin{aligned} |\tau v|_\circ^2 &= \sum_{k \in \mathbb{Z}} \|\iota_k \tau_k v_k\|_{(k)}^2 \leq \left( \sup_{k \in \mathbb{Z}} \|\iota_k \tau_k\| \right)^2 \sum_{k \in \mathbb{Z}} \|v_k\|_{(k)}^2 \\ &= \left( \sup_{k \in \mathbb{Z}} \|\iota_k \tau_k\| \right)^2 \|v\|_\circ^2. \end{aligned}$$

(surjectivity) Given  $\phi = \bigoplus_{k \in \mathbb{Z}} \phi_k \in \mathfrak{h}_\circ$ , let us define  $v := \bigoplus_{k \in \mathbb{Z}} v_k$  by  $v_k = \iota_k \phi_k \in \mathcal{H}_{(k)}$ . Then  $v \in \mathcal{H}_\circ$  by

$$\sum_{k \in \mathbb{Z}} \|v_k\|_{(k)}^2 = \sum_{k \in \mathbb{Z}} \|\iota_k \phi_k\|_{(k)}^2 = \sum_{k \in \mathbb{Z}} |\phi_k|_{(k)}^2 = |\phi|_\circ^2.$$

(density) Given  $v := \bigoplus_{k \in \mathbb{Z}} v_k \in \mathcal{H}$  and  $\epsilon > 0$ , let  $N_\epsilon \geq 0$  such that  $\sum_{|k| > N_\epsilon} \|v_k\|_k^2 \leq \epsilon/2$ . Since  $\mathcal{K}(\tau_k)$  is dense in  $\mathcal{H}_k$ , there exist  $v_{k,\epsilon} \in \mathcal{K}(\tau_k)$  such that  $\|v_k - v_{k,\epsilon}\|_k^2 \leq 2^{-|k|}(\epsilon/6)$ . Define  $v_\epsilon := \bigoplus_{|k| \leq N_\epsilon} v_{k,\epsilon}$ . Then  $v_\epsilon \in \mathcal{K}(\tau)$  and

$$\|v - v_\epsilon\|^2 \leq \sum_{|k| \leq N_\epsilon} \|v_k - v_{k,\epsilon}\|_k^2 + \frac{\epsilon}{2} \leq \frac{\epsilon}{6} \sum_{k \in \mathbb{Z}} 2^{-|k|} + \frac{\epsilon}{2} = \epsilon.$$

□

**Remark 2.2.** Notice that Theorem 2.1 holds true for any sequence of Hilbert spaces  $\mathcal{H}_{(k)}$ ,  $k \in \mathbb{N}$ , such that each  $\mathcal{H}_{(k)}$  is densely embedded in  $\mathcal{H}_k$ . However our hypotheses  $\mathcal{H}_{(k)} = \mathcal{D}(A_k)$  permits to show that it is always possible to find right inverses  $\iota_k$  such that hypothesis (2.1) is satisfied (see Lemma 2.3 below).

For any  $z \in \rho(A_k)$ , let us define the following bounded linear operators:

$$R_k(z) : \mathcal{H}_k \rightarrow \mathcal{H}_{(k)}, \quad R_k(z) := (-A_k + z)^{-1},$$

$$G_k(z) : \mathfrak{h}_k \rightarrow \mathcal{H}_k, \quad G_k(z) := (\tau_k R_k(\bar{z}))^*.$$

By resolvent identity one has

$$(2.2) \quad G_k(w) - G_k(z) = (z - w)R_k(w)G_k(z) = (z - w)R_k(z)G_k(w).$$

Now let us take  $z = \pm i$  in the above definitions and pose

$$R_k := (-A_k + i)^{-1}, \quad G_k := G_k(-i), \quad G_k^+ := G_k(i),$$

$$\Gamma_k(z) := \tau_k \left( \frac{G_k + G_k^+}{2} - G_k(z) \right).$$

Then  $z \mapsto \Gamma_k(z)$  is a Weyl function, i.e. it satisfied the identities

$$\Gamma_k(z) - \Gamma_k(w) = (z - w)G_k(\bar{w})^*G_k(z)$$

and

$$\Gamma_k(z)^* = \Gamma_k(\bar{z}).$$

Therefore the set

$$Z_k := \{z \in \rho(A_k) : 0 \in \rho(\Gamma_k(z))\}.$$

is not void:  $\mathbb{C} \setminus \mathbb{R} \subseteq Z_k$  (see e.g. [11, Theorem 2.1]).

Posing

$$\Gamma_k := \Gamma_k(-i),$$

one has the identities

$$(2.3) \quad G_k^+ - G_k = 2iR_kG_k,$$

$$(2.4) \quad G_k^*G_k = -i\Gamma_k$$

and so

$$(2.5) \quad \iota_k : \mathfrak{h}_k \rightarrow \mathcal{H}_{(k)}, \quad \iota_k := iR_kG_k\Gamma_k^{-1} = R_kG_k(G_k^*G_k)^{-1}.$$

is a linear bounded right inverse of  $\tau_k$ . Moreover, since  $R_k : \mathcal{H}_k \rightarrow \mathcal{H}_{(k)}$  is unitary w.r.t. the scalar product

$$\langle u_k, v_k \rangle_{(k)} := \langle (-A + i)u_k, (-A + i)v_k \rangle_k,$$

one has

$$(2.6) \quad \iota_k^* \iota_k = (G_k^*G_k)^{-1}.$$

**Lemma 2.3.** *Let  $\iota_k$  be defined as in (2.5). Then  $\|\iota_k \tau_k\| = 1$ .*

*Proof.* By (2.5) one has

$$\|\iota_k \tau_k v_k\|_{(k)} = \|G_k(G_k^*G_k)^{-1}G_k^*(-A_k + i)v_k\|_k.$$

Since the range of  $G_k$  is closed one has the decomposition  $\mathcal{H}_k = \mathcal{R}(G_k) \oplus \mathcal{K}(G_k^*)$  and so  $(-A_k + i)v_k = G_k\phi_k \oplus w_k$ . Therefore

$$\|\iota_k \tau_k v_k\|_{(k)} = \|G_k\phi_k\|_k \leq \|(-A_k + i)v_k\|_k = \|v_k\|_{(k)}.$$

If  $v_k = R_kG_k\phi_k$  then  $\|\iota_k \tau_k v_k\|_{(k)} = \|v_k\|_{(k)}$ . □

**Remark 2.4.** In the case there exists  $\lambda \in \cap_{k \in \mathbb{Z}} \rho(A_k) \cap \mathbb{R}$  the previous reasonings have the following variant. By (2.2) there follows

$$\begin{aligned} & G_k(-i)^* G_k(-i) \\ &= G_k(\lambda)^* (1 + (\lambda - i)R_k(i))(1 + (\lambda + i)R_k(-i))G_k(\lambda). \end{aligned}$$

and so

$$\begin{aligned} |G_k(-i)^* G_k(-i)\phi_k|_k &\leq \|1 + (\lambda - i)R_k(i)\|^2 |G_k(\lambda)^* G_k(\lambda)\phi_k|_k \\ &\leq \left(1 + \sqrt{1 + \lambda^2}\right)^2 |G_k(\lambda)^* G_k(\lambda)\phi_k|_k. \end{aligned}$$

Since  $G_k(-i)^* G_k(-i)$  is injective by (2.4), this shows that  $G_k(\lambda)^* G_k(\lambda)$  is injective. Since it is self-adjoint and its range is closed (since the range of  $G_k(\lambda)$  is closed),  $G_k(\lambda)^* G_k(\lambda)$  is a continuous bijection. Then

$$\iota_k := R_k G_k (G_k^* G_k)^{-1},$$

is a bounded right inverse of  $\tau_k$ , where here we pose

$$R_k := (-A_k + \lambda)^{-1}, \quad G_k := G_k(\lambda),$$

Moreover, by using the scalar product

$$\langle u_k, v_k \rangle_{(k)} := \langle (-A_k + \lambda)u_k, (-A_k + \lambda)v_k \rangle_k,$$

one gets

$$\iota_k^* \iota_k = (G_k^* G_k)^{-1}.$$

and, proceeding as in the proof of lemma 2.3,

$$\|\iota_k \tau_k\| = 1.$$

Theorem 2.1 has the following alternative version where one can still use the original trace space  $\mathfrak{h}$  as long as one regularizes the traces  $\tau_k$ :

**Theorem 2.5.** *Let us define  $r_k := (G_k^* G_k)^{1/2}$  and*

$$\tilde{\tau}_k : \mathcal{H}_{(k)} \rightarrow \mathfrak{h}_k, \quad \tilde{\tau}_k := r_k^{-1} \tau_k$$

*Then the linear map*

$$\tilde{\tau} : \mathcal{H}_\circ \rightarrow \mathfrak{h}, \quad \tilde{\tau}\left(\bigoplus_{k \in \mathbb{Z}} v_k\right) := \bigoplus_{k \in \mathbb{Z}} (\tilde{\tau}_k v_k)$$

*is continuous, surjective and its kernel  $\mathcal{K}(\tilde{\tau}) = \mathcal{K}(\tau)$  is dense in  $\mathcal{H}$ .*

*Proof.* The proof is the same as in Theorem 2.1. It suffices to notice that  $\tilde{\iota}_k := \iota_k r_k$  is the right inverse of  $\tilde{\tau}_k$  and that

$$(\tilde{\iota}_k)^* \tilde{\iota}_k = r_k \iota_k^* \iota_k r_k = r_k (G_k^* G_k)^{-1} r_k = 1.$$

□

**Remark 2.6.** Notice that in this section  $\mathbb{Z}$  can be replaced by any other denumerable set  $N$  and that we can replace  $[\cdot, \cdot]_{(k)}$  by a scalar product inducing an equivalent norm. Moreover, given a finite subset  $F \subset N$ , we can replace  $\mathfrak{h}_{(k)}$  by  $\mathfrak{h}_k$  for any  $k \in F$ .

### 3. APPLICATIONS AND EXAMPLES.

Let  $S_k$ ,  $k \in \mathbb{Z}$ , be the sequence of symmetric operators defined by  $S_k := A_k|_{\mathcal{H}(\tau_k)}$ , where  $A_k$  and  $\tau_k$  are defined as in the previous section. Then  $S := \bigoplus_{k \in \mathbb{Z}} S_k$  is a symmetric operator and  $S = A|_{\mathcal{H}(\tau)}$ , where  $A := \bigoplus_{k \in \mathbb{Z}} A_k$  and  $\tau := \bigoplus_{k \in \mathbb{Z}} \tau_k$  is defined as in Theorem 2.1. Here  $\tau_k$  is considered as a map on  $\mathcal{H}_{(k)}$  to  $\mathfrak{h}_{(k)}$ , so that when calculating the adjoint  $G_{(k)}(z)$  of  $\tau_k(R_k(\bar{z}))$  one gets

$$G_{(k)}(z) := G_k(z)\iota_k^* \iota_k.$$

Next Lemma shows that the direct sums  $\bigoplus_{k \in \mathbb{Z}} G_{(k)}(z)$  appearing in Theorem 3.2 below are well defined bounded operators:

**Lemma 3.1.**

$$\forall z \in \bigcap_{k \in \mathbb{Z}} \rho(A_k), \quad \sup_{k \in \mathbb{Z}} \|G_{(k)}(z)\| < +\infty.$$

*Proof.* By (2.2) one has, posing  $G_{(k)} := G_{(k)}(-i)$ ,

$$\|G_{(k)}(z)\| \leq \|1 - (i + z)R_k\| \|G_{(k)}\| \leq (2 + |z|) \|G_{(k)}\|.$$

By (2.6),

$$\|G_{(k)}\phi_k\|_k = \|G_k \iota_k^* \iota_k \phi_k\|_k = \langle \iota_k^* \iota_k \phi_k, G_k^* G_k \iota_k^* \iota_k \phi_k \rangle_k = |\phi_k|_{(k)}$$

and so

$$\|G_{(k)}\| = 1.$$

□

By Theorem 2.1 and by the results provided in [9, Theorem 2.2] and [11, Theorem 2.1] one gets the following

**Theorem 3.2.** *The set of self-adjoint extensions of  $S$  is parametrized by couples  $(\Pi, \Theta)$ , where  $\Pi$  is an orthogonal projection in  $\mathfrak{h}_\circ = \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_{(k)}$  and  $\Theta$  is a self-adjoint operator in the Hilbert space  $\text{Range}(\Pi)$ . Denoting by  $A_{\Pi, \Theta}$  the self-adjoint extension associated with  $(\Pi, \Theta)$  one has*

$$A_{\Pi, \Theta}(\bigoplus_{k \in \mathbb{Z}} v_k) = \bigoplus_{k \in \mathbb{Z}} (A_k v_k^\circ + (\text{Re}(z_\circ) G_{(k)}^\circ + i \text{Im}(z_\circ) G_{(k)}^\circ) \phi_k),$$

$$\mathcal{D}(A_{\Pi, \Theta}) = \left\{ \bigoplus_{k \in \mathbb{Z}} v_k \in \mathcal{H} : v_k = v_k^\circ + G_{(k)}^\circ \psi_k, \quad \bigoplus_{k \in \mathbb{Z}} v_k^\circ \in \bigoplus_{k \in \mathbb{Z}} \mathcal{D}(A_k), \right. \\ \left. \bigoplus_{k \in \mathbb{Z}} \phi_k \in \mathcal{D}(\Theta), \quad \Pi \left( \bigoplus_{k \in \mathbb{Z}} \tau_k v_k^\circ \right) = \Theta \left( \bigoplus_{k \in \mathbb{Z}} \phi_k \right) \right\}.$$

Moreover, for any  $z \in (\cap_{k \in \mathbb{Z}} \rho(A_k)) \cap \rho(A_{\Pi, \Theta})$ ,

$$(-A_{\Pi, \Theta} + z)^{-1} = \bigoplus_{k \in \mathbb{Z}} (-A_k + z)^{-1} \\ + \bigoplus_{k \in \mathbb{Z}} G_{(k)}(z) \Pi \left( \Theta + \Pi \bigoplus_{k \in \mathbb{Z}} \tau_k (G_{(k)}^\circ - G_{(k)}(z)) \Pi \right)^{-1} \Pi \bigoplus_{k \in \mathbb{Z}} G_{(k)}^*(z).$$

Here

$$G_{(k)}^\circ := \frac{1}{2}(G_{(k)}(z_o) + G_{(k)}(\bar{z}_o)), \quad G_{(k)}^\diamond := \frac{1}{2}(G_{(k)}(z_o) - G_{(k)}(\bar{z}_o))$$

and  $z_o \in \cap_{k \in \mathbb{Z}} \rho(A_k)$ .

**Remark 3.3.** By the definition of  $\mathcal{D}(A_{\Pi, \Theta})$  one has that  $A_{\Pi, \Theta}$  is a direct sum if and only if both  $\Pi$  and  $\Theta$  are direct sums.

In the case  $z_o \in \mathbb{R}$  one has  $G_{(k)}^\diamond = 0$  and

$$\tau_k(G_{(k)}^\circ - G_{(k)}(z)) = z G_k(z_o)^* G_k(z) \iota_k^* \iota_k = z G_k(z)^* G_k(z_o) \iota_k^* \iota_k$$

In the case  $0 \in \cap_{k \in \mathbb{Z}} \rho(A_k)$  one can take  $z_o = 0$ , so that

$$A_{\Pi, \Theta} \left( \bigoplus_{k \in \mathbb{Z}} v_k \right) = \bigoplus_{k \in \mathbb{Z}} A_k v_k^\circ.$$

**Remark 3.4.** Theorem 3.2 has an alternative version in the case one uses the trace map furnished by Lemma 2.5. In this case the extension parameter  $(\Pi, \Theta)$  is such that  $\Pi$  is an orthogonal projection in  $\mathfrak{h} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_k$  and  $\Theta$  is a self-adjoint operator in the Hilbert space associated with  $\Pi$ . The statement of Theorem 3.2 remains unchanged replacing  $\tau_k$  with  $\tilde{\tau}_k$  and  $G_{(k)}(z)$  with

$$\tilde{G}_k(z) := G_{(k)}(z) r_k = G_k(z) r_k^{-1} = (\tilde{\tau}_k R_k(\bar{z}))^*.$$

Also in this case the norm of  $\tilde{G}_{(k)}(z) : \mathcal{H}_k \rightarrow \mathfrak{h}_k$  is bounded uniformly in  $k \in \mathbb{Z}$  for any  $z \in \cap_{k \in \mathbb{Z}} \rho(A_k)$ .

**Remark 3.5.** By [10] and [11, Section 4], Theorem 3.2 and Lemma 2.5 provide results equivalent to the ones that can be obtained using Boundary Triplet Theory. Let us for simplicity take  $z_o = i$ . Then (see [10, Theorem 3.1])

$$\mathcal{D}(S_k^*) = \{v_k \in \mathcal{H} : v_k = v_k^\circ + G_k^\circ \phi_k, \quad v_k^\circ \in \mathcal{D}(A_k), \quad \phi_k \in \mathfrak{h}_k\}, \\ S_k^* : \mathcal{D}(S_k^*) \subseteq \mathcal{H}_k \rightarrow \mathcal{H}_k, \quad S_k^* v_k := A_k v_k + R_k G_k \psi_k,$$



and the triple  $\{\mathfrak{h}_k, \beta_{k,0}, \beta_{k,1}\}$ , where

$$\begin{aligned}\beta_{k,0} : \mathcal{D}(S_k^*) &\rightarrow \mathfrak{h}_k, & \beta_{k,0}v_k &:= \tau_k v_k^\circ, \\ \beta_{k,1} : \mathcal{D}(S_k^*) &\rightarrow \mathfrak{h}_k, & \beta_{k,1}v_k &:= \phi_k,\end{aligned}$$

is a boundary triple for  $S_k^*$ , i.e.  $\beta_{k,1}$  and  $\beta_{k,2}$  are surjective and satisfy the Green-type identity

$$(3.1) \quad \langle S_k^* u_k, v_k \rangle_k - \langle u_k, S_k^* v_k \rangle_k = [\beta_{1,k} u_k, \beta_{k,0} v_k]_k - [\beta_{k,0} u_k, \beta_{k,1} v_k]_k.$$

Moreover the Weyl function of  $S_k$  is

$$M_k(z) = \tau_k \left( \frac{G_k + G_k^+}{2} - G_k(z) \right)$$

(see [10, Theorem 3.1]). By (3.1) there follows that  $\{\mathfrak{h}_k, r_k \beta_{k,1}, r_k^{-1} \beta_{k,2}\}$ , where  $r_k$  is defined in Lemma 2.5, is a boundary triple for  $S_k^*$  as well with Weyl function  $r_k^{-1} M_k(z) r_k^{-1}$ .

By Lemma 2.5 and [10, Theorem 1.6] one gets

$$\begin{aligned}\mathcal{D}(S^*) &= \left\{ \bigoplus_{k \in \mathbb{Z}} v_k : v_k = v_k^\circ + \tilde{G}_k^\circ \phi_k, \bigoplus_{k \in \mathbb{Z}} v_k^\circ \in \bigoplus_{k \in \mathbb{Z}} \mathcal{D}(A_k), \bigoplus_{k \in \mathbb{Z}} \phi_k \in \mathfrak{h} \right\} \\ &\equiv \left\{ \bigoplus_{k \in \mathbb{Z}} v_k : v_k = v_k^\circ + G_k^\circ \psi_k, \bigoplus_{k \in \mathbb{Z}} v_k^\circ \in \bigoplus_{k \in \mathbb{Z}} \mathcal{D}(A_k), \bigoplus_{k \in \mathbb{Z}} r_k \psi_k \in \mathfrak{h} \right\}\end{aligned}$$

and the triple  $\{\mathfrak{h}, \tilde{\beta}_0, \tilde{\beta}_1\}$  is a boundary triple for  $S^* = \bigoplus_{k \in \mathbb{Z}} S_k^*$  with Weyl function  $\bigoplus_{k \in \mathbb{Z}} (r_k^{-1} M_k(z) r_k^{-1})$ , where

$$\begin{aligned}\tilde{\beta}_0 : \mathcal{D}(S^*) &\rightarrow \mathfrak{h}, & \tilde{\beta}_0 \left( \bigoplus_{k \in \mathbb{Z}} v_k \right) &:= \bigoplus_{k \in \mathbb{Z}} (r_k^{-1} \beta_{0,k}) \left( \bigoplus_{k \in \mathbb{Z}} v_k \right) \equiv \bigoplus_{k \in \mathbb{Z}} (r_k^{-1} \tau_k v_k^\circ), \\ \tilde{\beta}_1 : \mathcal{D}(S^*) &\rightarrow \mathfrak{h}, & \tilde{\beta}_1 \left( \bigoplus_{k \in \mathbb{Z}} v_k \right) &:= \bigoplus_{k \in \mathbb{Z}} (r_k \beta_{1,k}) \left( \bigoplus_{k \in \mathbb{Z}} v_k \right) \equiv \bigoplus_{k \in \mathbb{Z}} (r_k \psi_k) \equiv \bigoplus_{k \in \mathbb{Z}} \phi_k.\end{aligned}$$

Let us notice that the norm of  $r_k^{-1} \tau_k \equiv \tilde{\tau}_k : \mathcal{H}_{(k)} \rightarrow \mathfrak{h}_k$  is bounded uniformly in  $k \in \mathbb{Z}$  by Lemma 2.5, and so  $\tilde{\beta}_0$  is well defined.  $\tilde{\beta}_1$  is well defined as well by the definition of  $\mathcal{D}(S^*)$ .

In conclusion this provides results on direct sums of regularized boundary triplets of the kind recently obtained in [8, Section 5], [7, Section 3], [5, Section 2].

**Example 1.** Given  $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ ,  $x_n < x_{n+1}$ , let  $\mathcal{H}_n := L^2(I_n)$ ,  $n \geq 0$ , where  $I_0 = (-\infty, x_1]$  and  $I_n = [x_n, x_{n+1}]$ ,  $n \in \mathbb{N}$ . Let

$$\begin{aligned}A_n : \mathcal{D}(A_n) &\subset L^2(I_n) \rightarrow L^2(I_n), & A_n u &= u'', & n \geq 0, \\ \mathcal{D}(A_0) &:= \{u \in L^2(I_0) \cap C^1(I_0) : u'' \in L^2(I_0), u(x_1) = 0\}, \\ \mathcal{D}(A_n) &:= \{u \in C^1(I_n) : u'' \in L^2(I_n), u(x_n) = u(x_{n+1}) = 0\}, & n \in \mathbb{N},\end{aligned}$$

$$\begin{aligned}\tau_0 : \mathcal{H}_{(0)} &\rightarrow \mathbb{C}, \quad \tau_0 u := -u'(x_1). \\ \tau_n : \mathcal{H}_{(n)} &\rightarrow \mathbb{C}^2, \quad \tau_n u := (u'(x_n), -u'(x_{n+1})) \quad n \in \mathbb{N}.\end{aligned}$$

For any  $n \geq 0$ , the map  $\tau_n$  is continuous, surjective and has a kernel dense in  $L^2(I_n)$ .

By Remark 2.6 we can suppose  $n \neq 0$  and, since  $0 \in \cap_{n>0} \rho(A_n)$ , we can use the results provided in Remark 2.4 with  $\lambda = 0$ .

The kernel of  $(-A_n)^{-1}$ ,  $n > 0$ , is given by

$$\begin{aligned}K_n(x, y) \\ = \frac{1}{d_n} ((x_{n+1} - x)(y - x_n)\theta(x - y) + (x - x_n)(x_{n+1} - y)\theta(y - x)),\end{aligned}$$

where  $\theta$  denotes Heaviside's function and  $d_n := x_{n+1} - x_n$ . Therefore

$$(G_n \xi)(x) = \frac{1}{d_n} (\xi_1(x_{n+1} - x) + \xi_2(x - x_n)), \quad \xi \equiv (\xi_1, \xi_2),$$

$$G_n^* u \equiv \frac{1}{d_n} \left( \int_{x_n}^{x_{n+1}} (x_{n+1} - x)u(x) dx, \int_{x_n}^{x_{n+1}} (x - x_{n+1})u(x) dx \right)$$

and so by straightforward calculations one gets that  $G_n^* G_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  corresponds to the positive-definite matrix

$$G_n^* G_n \equiv d_n \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}.$$

In conclusion on  $\mathfrak{h}_{(n)} = \mathbb{C}^2$  we can put the equivalent scalar product

$$[\xi, \zeta]_{(n)} := \frac{\xi \cdot \zeta}{d_n}.$$

Hence, by Theorem 2.1, denoting by  $\ell_d^2(\mathbb{N})$  the weighted  $\ell^2$ -space

$$\ell_d^2(\mathbb{N}) := \left\{ \{s_n\}_{n \in \mathbb{N}} : \sum_{n \in \mathbb{N}} \frac{|s_n|^2}{d_n} < +\infty \right\},$$

one gets that

$$(3.2) \quad \tau := \tau_0 \oplus \left( \bigoplus_{n \in \mathbb{N}} \tau_n \right) : \mathcal{H}_0 \oplus \left( \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{(n)} \right) \rightarrow \mathbb{C} \oplus \ell_d^2(\mathbb{N}) \oplus \ell_d^2(\mathbb{N})$$

is continuous, surjective and has a kernel dense in  $L^2(-\infty, x_\infty)$ ,  $x_\infty := \sup_{n \in \mathbb{N}} x_n$ . Notice that  $\ell_d^2(\mathbb{N}) = \ell^2(\mathbb{N})$  if and only if

$$0 < d_* := \inf_{n \in \mathbb{N}} d_n \leq d^* := \sup_{n \in \mathbb{N}} d_n < +\infty.$$

By using Theorem 3.2 with trace map  $\tau$  defined in (3.2), one gets the same kind of self-adjoint extensions given in [7] (the case in which  $0 < d_* \leq d^* < +\infty$  has been studied in [6]). Such extensions describe one-dimensional Schrödinger operators in  $L^2(-\infty, x_\infty)$  with  $\delta$

and  $\delta'$  interactions supported on the discrete set  $X = \{x_n\}_{n \in \mathbb{N}}$ . These operators have been studied in [1, Chapters III.2 and III.3], when  $0 < d_* \leq d^* < +\infty$  and  $x_\infty = +\infty$ , and in [7] when  $d^* < +\infty$ . Analogous considerations, with  $A_n$  given by the one-dimensional Dirac operator with Dirichlet boundary conditions on the interval  $I_n$ , lead to self-adjoint extension describing one-dimensional Dirac operators with  $\delta$  and  $\delta'$  interactions on the discrete set  $X = \{x_n\}_{n \in \mathbb{N}}$  (see [1, Appendix J], for the case  $X$  in which is a finite set and [5] for the general case).

**Example 2.** At first let us check that applying Theorem 2.1 to the example given in the introduction one gets the right trace space  $\mathfrak{h}_\circ = h^{\frac{1}{2}}(\mathbb{Z})$ . Hence here  $A_k = d_0^2 - k^2$ . By Remark 2.6 we can suppose  $k \neq 0$  and, since  $0 \in \cap_{k \in \mathbb{Z} \setminus \{0\}} \rho(A_k)$ , we can use the results provided in Remark 2.4 with  $\lambda = 0$ . Since the kernel of  $(-d_0^2 + z^2)^{-1}$ ,  $\operatorname{Re}(z) > 0$ , is given by

$$K(z; x, y) = \frac{e^{-z|x-y|} - e^{-z(x+y)}}{2z},$$

one easily gets

$$G_k^* \equiv \hat{\gamma}_0(-d_0^2 + k^2)^{-1} : L^2(\mathbb{R}_+) \rightarrow \mathbb{C}, \quad G_k^* f = \int_0^\infty e^{-|k|x} f(x) dx$$

and so  $G_k^* G_k : \mathbb{C} \rightarrow \mathbb{C}$  is given by the multiplication by the real number

$$G_k^* G_k \equiv \int_0^\infty e^{-2|k|x} dx = \frac{1}{2|k|}.$$

Therefore  $\mathfrak{h}_{(k)} = \mathbb{C}$  is equipped with the scalar product

$$[\xi, \zeta]_{(k)} := |k| \xi \cdot \zeta$$

and so

$$\mathfrak{h}_\circ = \mathbb{C} \oplus \left( \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{h}_{(k)} \right) = \left\{ \{s_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{k \in \mathbb{Z}} |k| |s_k|^2 < +\infty \right\}.$$

By using Theorem 3.2 with trace map

$$\gamma_0 = \bigoplus_{k \in \mathbb{Z}} \hat{\gamma}_0 : \bigoplus_{k \in \mathbb{Z}} \mathcal{D}_0 \rightarrow h^{\frac{1}{2}}(\mathbb{Z}),$$

then one can determine all self-adjoint extensions of the minimal Laplacian on  $\mathbb{M}_0$ .

Such an example can be generalized in the following way: let  $\mathbb{M}_\alpha$  be  $\mathbb{R}_+ \times \mathbb{T}$  endowed with the singular/degenerate Riemannian metric

$$g_\alpha(x, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2\alpha} \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

The Riemannian volume form corresponding to  $g_\alpha$  is  $d\omega = x^{-\alpha} dx d\theta$  and so we denote by  $L^2(\mathbb{M}_\alpha)$  be the Hilbert space

$$L^2(\mathbb{M}_\alpha) := \{u : \mathbb{R}_+ \times \mathbb{T} \rightarrow \mathbb{C} : \int_0^{2\pi} \int_0^\infty |u(x, \theta)|^2 x^{-\alpha} dx d\theta < +\infty\}.$$

In [4] it is shown that the minimal realization

$$\Delta_\alpha^{\min} : C_c^\infty(\mathbb{M}_\alpha) \subset L^2(\mathbb{M}_\alpha) \rightarrow L^2(\mathbb{M}_\alpha)$$

of the Laplace-Beltrami operator

$$(3.3) \quad \Delta_\alpha := \frac{\partial^2}{\partial x^2} - \frac{\alpha}{x} \frac{\partial}{\partial x} + x^{2\alpha} \frac{\partial^2}{\partial \theta^2}$$

corresponding to  $g_\alpha$  is essentially self-adjoint whenever  $\alpha \notin (-3, 1)$ , has deficiency indices  $(1, 1)$  whenever  $\alpha \in (-3, -1]$  and has infinite deficiency indices whenever  $\alpha \in (-1, 1)$ . Therefore, in order to determine and then study all self-adjoint realizations of  $\Delta_\alpha^{\min}$ ,  $-1 < \alpha < 1$ , by Theorem 3.2 one needs to characterize the range space of the trace map

$$\gamma_\alpha u(\theta) := \lim_{x \downarrow 0} x^{-\alpha} \frac{\partial u}{\partial x}(x, \theta)$$

acting on function in the domain of the Friedrichs extensions  $\Delta_\alpha^D$  (corresponding to Dirichlet boundary conditions at  $\mathbb{T}$ ) of  $\Delta_\alpha^{\min}$  (see [12]). Let us sketch here a proof in the case  $0 < \alpha < 1$ , referring to [12] for more details and for the (more involved but still using Theorem 2.1) proof that holds in the case  $-1 < \alpha < 1$ .

By partial Fourier transform one gets

$$L^2(\mathbb{M}_\alpha) = \bigoplus_{k \in \mathbb{Z}} L_w^2(\mathbb{R}_+), \quad \Delta_\alpha^D = \bigoplus_{k \in \mathbb{Z}} (d_\alpha^2 - k^2 q_\alpha),$$

where  $L_w^2(\mathbb{R}_+)$  is the weighted  $L^2$  space

$$L_w^2(\mathbb{R}_+) := \{f : \mathbb{R}_+ \rightarrow \mathbb{C} : \int_0^\infty |f(x)|^2 x^{-\alpha} dx < +\infty\},$$

and

$$(d_\alpha^2 - k^2 q_\alpha) : \mathcal{D}_{\alpha, k} \subset L_w^2(\mathbb{R}_+) \rightarrow L_w^2(\mathbb{R}_+),$$

$$d_\alpha^2 f(x) := f''(x) - \frac{\alpha}{x} f'(x), \quad q_\alpha(x) = x^{2\alpha},$$

$$\mathcal{D}_{\alpha, k} := \{f \in L_w^2(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}_+}) : (d_\alpha^2 - k^2 q_\alpha) f \in L_w^2(\mathbb{R}_+), f(0) = 0\}.$$

By Remark 2.6 we can suppose  $k \neq 0$  and, since  $0 \in \bigcap_{k \in \mathbb{Z} \setminus \{0\}} \rho(A_k)$ ,  $A_k = d_\alpha^2 - k^2 q_\alpha$ , whenever  $0 < \alpha < 1$ , we can use the results provided

in Remark 2.4 with  $\lambda = 0$ . Since  $f_\xi \equiv G_k \xi$  solves the boundary value problem

$$\begin{cases} f_\xi''(x) - \frac{\alpha}{x} f_\xi'(x) - k^2 x^{2\alpha} f_\xi = 0 \\ f_\xi(0) = \xi, \end{cases}$$

one gets

$$(G_k \xi)(x) = \xi \exp\left(-\frac{|k|x^{\alpha+1}}{\alpha+1}\right).$$

Therefore  $G_k^* G_k : \mathbb{C} \rightarrow \mathbb{C}$  is given by the multiplication by the real number

$$G_k^* G_k \equiv \int_0^\infty e^{-2 \frac{|k|x^{\alpha+1}}{\alpha+1}} x^{-\alpha} dx = |k|^{\frac{\alpha-1}{\alpha+1}} \int_0^\infty e^{-2 \frac{x^{\alpha+1}}{\alpha+1}} x^{-\alpha} dx$$

and so  $\mathfrak{h}_{(k)} = \mathbb{C}$  is equipped with the scalar product

$$[\xi, \zeta]_{(k)} := |k|^{\frac{1-\alpha}{1+\alpha}} \xi \cdot \zeta.$$

Thus by Theorem 2.1 the range space of  $\gamma_\alpha$  (i.e. the defect space of  $\Delta_\alpha^{\min}$ ) is given by the fractional Hilbert-Sobolev space

$$H^s(\mathbb{T}) \simeq h^s(\mathbb{Z}) := \left\{ \{s_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{k \in \mathbb{Z}} |k|^{2s} |s_k|^2 < +\infty \right\},$$

where  $s = \frac{1}{2} - \frac{\alpha}{1+\alpha}$ .

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